ON GRAPHS WITH MINIMUM ELONGATION DIAMETER

by

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On Graphs With Minimum Elongation Diameter

Let A= {1, 2,...,n} be a finite set consisting of n elements (called vertices) and let E be the set of all possible lines (called edges) joining pairs of distinct points belonging to the set A. The pair (A,F), $\phi \subset F \subset E$ is called a finite graph of order n, and is denoted by G(A,F) or $G_{n,N}$ (or simply by G whenever both n and N are fixed) where n is the number of elements in set A and N is the number of elements in subset F. We assume that a graph cannot have multiple edges or loops (that is, edges joining a point to itself). If a graph has no edges at all, it is called as a null graph; if it has all possible edges, n(n-1)/2, it is called a complete graph. A path of length L is a sequence of edges $\{e_i = (i-1, i), i=1, 2,\dots,L\}$, where e_{i-1} and e_i always have a common vertex, each e_i appears once and only once, but the vertices may appear more than once. A path is said to be a simple path if none of its vertices are transversed more than once. The length of the shortest simple path between i and j is called the distance of i and j, and is denoted by d(i,j); the diameter of a graph G is the maximum of d(i,j) over all pairs (i,j), $i \neq j$, and is denoted by $\delta(G)$. The length of the longest simple path between i and j is called the elongation of i and j, and is denoted by e(i,j); the elongation diameter of a graph G is the maximum of e(i,j) over all pairs (i,j), $i \neq j$, and is denoted by e(G). Finally, a graph G is said to be connected if for every pair of points (i,j) in A there exists a path between i and j.

In this paper, we solve the problem of characterizing connected graphs with a fixed number of edges and vertices, which have (a) minimum elongation diameter, and (b) minimum diameter.

Elongation Diameter

Let $\mathfrak{G}$ be the class of all connected finite graphs with n vertices and N edges; that is

$$\mathfrak{G} = \{G = G_{n,N} : G \text{ connected}\}, \; n \geq 2; \quad n-1 \leq N \leq \frac{n(n-1)}{2}.$$

Let $\mathfrak{G}^* \subset \mathfrak{G}$ be a subclass of connected graphs such that if G^* belongs to $\mathfrak{G}^*$ and G belongs to $\mathfrak{G}$, then $e(G^*) \leq e(G)$. We wish to characterize $\mathfrak{G}^*$. In other words, we wish to find a collection of graphs which are connected, have fixed number of vertices and edges, and for which the maximum distance between any pair of its vertices is as small as possible.

Let us partition the vertex set A of a graph of order n into two subsets A_1 and A_2 such that through each vertex in A_1 there are exactly n-1 edges (to the other n-1 vertices in A), while the number of edges through any vertex in A_2 is strictly less than n-1. Clearly $A_1 \cup A_2 = A$, and $A_1 \cap A_2 = \phi$.

A graph G of order n on set A is said to be exact k-complete, $k \geq 1$, if subset A_1 has k vertices, and none of the vertices in subset A_2 are joined by means of edges to each other; such a graph is denoted by G(n,k). We call the points in subset A_1 complete vertices of graph G and those in subset A_2 as incomplete vertices of graph G(n,k). A graph G of order n on set A is said to be plus k-complete if it is exact k-complete and there is exactly one incomplete vertex which is joined by means of edges to d of the remaining n-k-1 incomplete vertices, and $1 \leq d < n-k-1$; such a graph is denoted by $G(n,k^+)$. Finally, a graph G is said to be k-complete if it is either exact k-complete or plus k-complete. We note that any of the three kinds of k-complete graphs mentioned above are necessarily connected.

Let $\mathcal{C}, \mathcal{C}', \mathcal{C}''$ denote the classes of all k-complete, exact k-complete, and plus k-complete graphs of order n. To avoid trivial exceptions later, we take $n \geq 3$. We note that $\mathcal{C}' \cup \mathcal{C}'' = \mathcal{C}$, $\mathcal{C}' \cap \mathcal{C}'' = \phi$, and $\mathcal{C}, \mathcal{C}', \mathcal{C}'' \subset \mathcal{G}$.

We now state the main result of this paper: a k-complete graph has minimum elongation diameter. Formally,

Theorem 1. Let G^* be a k-complete graph of order n and G any other graph of the same order and with the same number of edges. Then $e(G^*) \leq e(G)$, where G^* belongs to $\mathcal{C}$ and G to $\mathcal{G}$.

The proof of this theorem requires several results, regarding the k-complete graphs, which are stated in the form of lemmas 1.1 through 1.7.

Lemma 1.1. $e\big(G(n,k)\big) < n-1$ if, and only if, $2k < n-1$.

Proof: Let P be that path in $G(n,k)$ whose length is maximum, that is equal to $e(G)$. Clearly if P is the longest path in $G(n,k)$ then it must contain each vertex of A_1.

If the length of P is less than n-1, then by construction of subsets A_1 and A_2 the path P must contain each vertex in A_1 and fail to contain at least one vertex from A_2; otherwise the length of P equals n-1. For the same reason no two vertices which are adjacent in P can be in A_1; we note that no two adjacent vertices can be in A_2 by definition. Furthermore such a path P can neither start nor end with a vertex in A_1; for then P transverses all of the vertices in $G(n,k)$, and the length of the path is once again n-1. Hence k must be less than $(n-1)/2$, that is $2k < n-1$.

Conversely if $2k < n-1$, then $e(G(n,k))$ must be less than n-1, since then the maximum number of vertices P may contain cannot exceed 2k+1 so that the length of P cannot exceed 2k.

As a consequence of the above lemma, we remark that given a pair of vertices i and j belonging to a graph $G(n,k)$, for which $e(G(n,k)) < n-1$, the longest path between i and j is such that no two adjacent vertices in the path belong to the same subset A_s $(s = 1,2)$ of the set A.

The following lemma is an obvious corollary of the preceding lemma, but is stated separately here due to its repeated use in later results.

<u>Lemma 1.2</u>. Given $G(n,k)$, $2k \geq n-1$ implies $e(G(n,k)) = n-1$, and $2k < n-1$ implies $e(G(n,k)) = 2k$.

Proofs of lemmas 1.3, 1.4 and 1.5 are rather obvious and as such are omitted:

Lemma 1.3. If a graph $G_{n,N}$ belonging to $\mathcal{G}$ contains an exact k-complete graph $G(n,k)$ with $N-1$ edges then $e(G_{n,N}) \leq 2k + 1$.

Lemma 1.4. If a graph $G_{n,N}$ belonging to $\mathcal{G}$ contains an exact k-complete graph $G(n,k)$ with $N-2$ edges then $e(G_{n,N}) \leq 2k + 2$.

Lemma 1.5. Let the graph $G(n,k^+)$ have N edges. Then

$$e\left(G(n,k^+)\right) \leq \begin{cases} 2k+1, & \text{if } N = kn - \frac{k(k+1)}{2} + 1 \\ 2k+2, & \text{if } kn - \frac{k(k+1)}{2} + 1 < N < (k+1)n - \frac{(k+1)(k+2)}{2} \end{cases}$$

Lemma 1.6. Let $G_{n,N}$ be a graph belonging to $\mathcal{G}$ such that $N = kn - \frac{k(k+1)}{2}$, and let $G(n,k)$ be an exact k-complete graph with the same number, N, of edges. Then

$$e\left(G(n,k)\right) \leq e(G_{n,N}).$$

Proof: First we note that for a connected graph G, $e(G)$ cannot be equal to one; therefore, the elongation diameter $e(G) \geq 2$. Next we easily see that for $G(n,1)$, $e(G) = 2$. Mathematical induction on $k \geq 2$ completes the proof.

Lemma 1.7. Let $G_{n,N}$ be a graph belonging to $\mathcal{G}$ such that $kn - \frac{k(k+1)}{2} < N < (k+1)n - \frac{(k+1)(k+2)}{2}$, and let $G(n,k^+)$ be a plus k-complete graph with N edges. Then $e\left(G(n,k^+)\right) \leq e(G_{n,N})$.

Proof: Lemmas 1.5 and 1.6.

Theorem 1 is now clearly seen as a combined restatement of lemmas 1.6 and 1.7.

We also make the following trivial observation:

Theorem 1'. Let $G_{n,N}$ be a graph in $\mathcal{G}$ such that

$$N \geq \begin{cases} (3n^2 - 2n)/8, & \text{if } n \text{ is even} \\ (3n^2 - 4n + 1)/8, & \text{if } n \text{ is odd.} \end{cases}$$

Then $e(G_{n,N}) = n-1$.

Minimum Diameter

The following result on minimum diameter of a connected graph is obtained as a consequence of our preceding work on elongation diameters.

Theorem 2. Let $G_{n,N}$ be a graph in $\mathcal{G}$. Then (i) $\delta(G_{n,N}) = 1$ if, and only if, $N = \frac{n(n-1)}{2}$ (that is $G_{n,N}$ is complete), (ii) If $N \neq \frac{n(n-1)}{2}$, and $G^*_{n,N}$ is a graph such that it has a subgraph $G_{n,n-1}$ which is exact 1-complete, that is $G_{n,n-1} = G(n,1)$ with n-1 edges, then $\delta(G^*_{n,N}) \leq \delta(G_{n,N})$.

Reference

Ore, O., Theory of Graphs, American Mathematical Society Colloquium Publication, 38, Providence, 1962.

Appendix

Figure 1. Example of an exact k-complete graph; $n=7$, $k=2$, $N=11$.

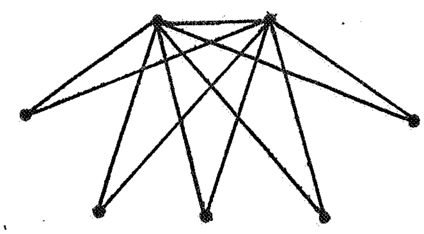


Figure 2. Example of a plus k-complete graph; $n=7$, $k=2$, $N=14$.

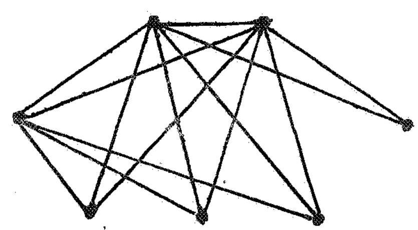


Figure 3. Example of a connected graph $G_{n,N}$ with a subgraph $G_{n,N-1}$ which is exact k-complete; $n=7$, $k=2$, $N=12$.

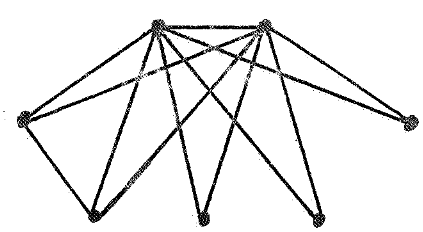
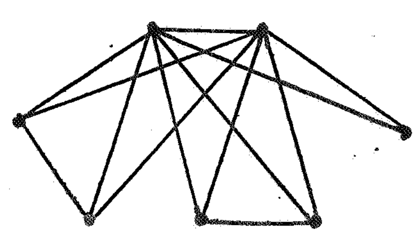


Figure 4. Example of a connected graph $G_{n,N}$ with a subgraph $G_{n,N-2}$ which is exact k-complete; $n=7$, $k=2$, $N=13$.



An Addendum to Technical Report No. 15*

In the technical report No. 15 entitled "On Graphs with Minimum Elongation Diameter", we find that the concept of k-complete graphs" introduced by us becomes a little clearer if we illustrate the definitions by means of examples given in figures 1 and 2 of next page (page no. 7); figures 3 and 4 help in understanding more clearly lemmas 1.3 and 1.4.

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